数理解析研究所講究録
1012巻 1997年 74-77

ON MEROMORPHIC CONVEX AND STARLIKE FUNCTIONS

By

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Abstract

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and

$$1 + \mathrm{Re}\frac{zf''(z)}{f'(z)} > 0 \qquad in \quad |z| < 1.$$

Then it is well known that [1, 3]

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \qquad in \quad |z| < 1.$$

Corresponding the above theorem, if $f(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$ is analytic in the punctured disk $0 < |z| < 1$ and

$$\mathrm{Re}[-(1 + \frac{zf''(z)}{f'(z)})] > 0 \qquad in \quad |z| < 1,$$

then there exists no positive constant $\alpha > 0$ for which

$$\mathrm{Re}[-\frac{zf'(z)}{f(z)}] > \alpha \qquad in \quad |z| < 1.$$

1. Introduction.

Let Σ denote the class of function of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$.

A function $f(z) \in \Sigma$ is called meromorphic starlike of order α $(0 \leq \alpha < 1)$ if

$$\mathrm{Re}[-\frac{zf'(z)}{f(z)}] > \alpha$$

for all $z \in U = \{z : |z| < 1\}$.

We denote by $\Sigma^*(\alpha)$ the subclass of Σ consisting of functions which are meromorphic starlike of order α in U.

Further, a function $f \in \Sigma$ is called meromorphic convex of order α $(0 \leq \alpha < 1)$ if

$$\mathrm{Re}[-(1 + \frac{zf''(z)}{f'(z)})] > \alpha$$

for all $z \in U$.

We denote by $\Sigma_c(\alpha)$ the subclass of Σ consisting of functions which are meromorphic convex of order α in U.

2. Preliminaries.

Lemma 1. [1, Theorem 1] Let $p(z)$ be regular in U, $p(0) = 1$ and suppose that there exists a point $z_0 \in U$ such that

$$\mathrm{Re}p(z) > 0 \qquad for \quad |z| < |z_0|,$$

$$\mathrm{Re}p(z_0) = 0 \qquad and \quad p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and $|k| \geq 1$.

Lemma 2. Let $p(z)$ be regular in U, $p(0) = 1$ and

(1) $$\mathrm{Re}(p(z) - \frac{zp'(z)}{p(z)}) > 0 \qquad (z \in U),$$

then

$$\mathrm{Re}p(z) > 0 \qquad (z \in U).$$

Then this result is sharp for the function $p(z) = \frac{1+z}{1-z}$.

Proof. First, we want to prove $p(z) \neq 0$ $(z \in U)$.

If $p(z)$ has a zero of order n $(n \geq 1)$ at a point z_0 $(z_0 \neq 0)$, then $p(z)$ can be written as $p(z) = (z - z_0)^n g(z)$ $\quad (g(z_0) \neq 0$, $g(z)$ is regular in U), and it follows that

$$p(z) - \frac{zp'(z)}{p(z)} = (z - z_0)^n g(z) - \frac{nz}{z - z_0} - \frac{zg'(z)}{g(z)}.$$

When z approaches to z_0 on the line segment satisfying the conditions $argz = argz_0 = \theta$ and $|z_0| < |z| < 1$, we have

$$\begin{aligned} &\lim_{\substack{z \to z_0 \\ argz = argz_0, |z_0| < |z| < 1}} \mathrm{Re}(p(z) - \frac{zp'(z)}{p(z)}) \\ = &\lim_{\substack{z \to z_0 \\ argz = argz_0, |z_0| < |z| < 1}} \mathrm{Re}((z - z_0)^n g(z) - \frac{nz}{z - z_0} - \frac{zg'(z)}{g(z)}) \\ = &\text{ negative infinite real value,} \end{aligned}$$

because we have

$$\lim_{\substack{z\to z_0 \\ \arg z=\arg z_0, |z_0|<|z|<1}} \left(\arg\left(-\frac{nz}{z-z_0}\right)\right)$$
$$= \lim_{\substack{z\to z_0 \\ \arg z=\arg z_0, |z_0|<|z|<1}} (\arg(-1)+\arg nz-\arg(z-z_0))$$
$$= \pi+\theta-\theta=\pi.$$

This result contradicts (1) .
Therefore we have

$$p(z)\neq 0 \quad (z\in U).$$

If there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z)>0 \qquad for \quad |z|<|z_0|,$$

$$\operatorname{Re} p(z_0)=0 \qquad and \quad p(z_0)\neq 0,$$

then from Lemma 1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)}=ik$$

where k is real and $|k|\geq 1$.

For the case $p(z_0)=ia$ $(a>0)$, we have

$$\operatorname{Re}\left(\frac{z_0 p'(z_0)}{p(z_0)}-p(z_0)\right)=\operatorname{Re}(ik-ia)=0.$$

This contradicts our assumption.

For the case $p(z_0)=-ia$ $(a>0)$, applying the same method as the above, we have

$$\operatorname{Re}\left(\frac{z_0 p'(z_0)}{p(z_0)}-p(z_0)\right)=0.$$

This contradicts our assumption.
Therefore we complete our proof.
The result is sharp for the function $p(z)=\frac{1+z}{1-z}$.

3. Main result.

Theorem. If $f(z)\in\Sigma_c(0)$, then $f(z)\in\Sigma^*(0)$, and there exists no positive constant $\alpha>0$ such that $\Sigma_c(0)\subset\Sigma^*(\alpha)$.

Proof. Setting

$$p(z)=-\frac{zf'(z)}{f(z)},$$

then we have $p(0)=1$ and

$$-\left(1+\frac{zf''(z)}{f'(z)}\right)=p(z)-\frac{zp'(z)}{p(z)}.$$

From the assumption of theorem, we have

$$\mathrm{Re}[-(1+\frac{zf''(z)}{f'(z)})] = \mathrm{Re}[p(z) - \frac{zp'(z)}{p(z)}] > 0 \qquad in \quad U,$$

then from Lemma 2, we have

$$\mathrm{Re}(-\frac{zf'(z)}{f(z)}) = \mathrm{Re}p(z) > 0 \qquad in \quad U.$$

Next, we prove that there exists no positive constant $\alpha > 0$ such that $\Sigma_c(0) \subset \Sigma^*(\alpha)$. Because the extremal function of Lemma 2 is

$$p(z) = \frac{1+z}{1-z},$$

so we put

$$-\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}.$$

Then by a brief calculation, we have

$$\frac{f'(z)}{f(z)} = -\frac{1}{z} - \frac{2}{1-z}.$$

Adding $1/z$ to both sides and integrating from zero to z $(0 < |z| < 1)$, we have

$$\int_0^z (\frac{1}{z} + \frac{f'(z)}{f(z)})dz = -\int_0^z \frac{2}{1-z}dz,$$

and it follows that

$$f(z) = \frac{(1-z)^2}{z}.$$

This function belong to $\Sigma_c(0)$ and $\Sigma^*(0)$ but there exists no positive constant $\alpha > 0$ for which $f(z) \in \Sigma^*(\alpha)$.

References

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